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# A unified thermodynamic picture of Hořava-Lifshitz black hole in arbitrary space time

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**ABSTRACT:** In this paper, we analyze the complete thermodynamic and phase transition phenomena of a black hole solution in Hořava-Lifshitz gravity in arbitrary space time. Nature of phase transition is studied using geometrothermodynamic and Ehrenfest's scheme of standard thermodynamics. We analytically check the Ehrenfest's equations near the critical point, which is the point of divergence in the heat capacity. Our analysis reveals that this black hole exhibits a second order phase transition.

**KEYWORDS:** Models of Quantum Gravity, Black Holes, Classical Theories of Gravity

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## 1 Introduction

Black hole thermodynamics has been a fascinating topic of study since black holes were identified as thermodynamic objects with both temperature and entropy [1, 2]. Thermodynamic properties of black holes have been studied during all these years. During this period, it became well known that the black hole spacetime can possess phase structures along with the standard thermodynamic variables like temperature, entropy etc. Hence it causes to believe in the existence of a complete analogy between black hole system and non-gravitational thermodynamic systems. Various black hole thermodynamic variables and their properties have been extensively studied. In 1983, Hawking and Page [3] discovered the phase transition phenomena in the Schwarzschild AdS background. This became a turning point in the study of black hole phase transition. After this, many studies have been done in this regard [62–87].

Recently geometric method has been identified as a convenient tool to study the thermodynamics and the corresponding phase transition structures of black holes. Various investigations are also done by incorporating this idea from information geometry to the study of black hole thermodynamics [4–8]. Riemannian geometry to the equilibrium space was first introduced by Weinhold and Ruppeiner. In 1976 Weinhold [9–13] proposed a metric, as the Hessian of the internal energy, given as  $g_{ij}^W = \partial_i \partial_j U(S, N^r)$ . Later in 1979, Ruppeiner [14] introduced another metric as the Hessian of entropy, as  $g_{ij}^R = -\partial_i \partial_j S(M, N^r)$ . This Ruppeiner metric is conformally equivalent to Weinhold’s metric and the geometry that can be obtained from these two methods are related through the relation where [15, 16],  $ds_R^2 = \frac{1}{T} ds_W^2$ . Since these matrices depend on the choice of thermodynamic potentials, they are not Legendre invariant. The results obtained with the above two metrics are found to be consistent with the systems like ideal classical gas, multicomponent ideal gas, ideal quantum gas, one-dimensional Ising model, van der Waals model etc, [17–27]. But these two metrics fail in explaining the thermodynamic properties and they lead to many puzzling situations. By incorporating the idea of Legendre invariance, Quevedo et al. [48–50]

proposed a new geometric formalism, known as the Geometrothermodynamics. The metric structure related with geometrothermodynamics can give well explanations for different behaviors of black hole thermodynamic variables. This method seizes the exact phase structure of black hole systems.

On the other hand, classical thermodynamics can be applied to black hole systems directly to study the phase structure. In classical thermodynamics first order phase transitions satisfy Clausius-Clapeyron equation, while second order phase transitions satisfy Ehrenfest equations. Recently Banerjee et al. [53–58] developed a new scheme to study the phase transition in black holes based on Ehrenfest equations by considering the analogy between thermodynamic variables and black hole parameters. This Ehrenfest scheme provides a unique way to classify the nature of phase transitions in black hole systems. If two Ehrenfest relations are satisfied by a black hole system, then the corresponding phase transition can be identified as second order in nature. Even if it is not second order, we can find the deviation from second order by defining Ehrenfest’s relation as quantified by defining a parameter called the Prigogine-Defay(PD) ratio [59–61]. Interestingly many calculations show that there are black hole systems whose phase transitions lie within the bound conforming to a glassy phase transition.

Hořava-Lifshitz theory is an anisotropic and a non-relativistic renormalizable theory of gravity at a Lifshitz point, which can be treated as a good candidate for the study of quantum field theory of gravity, and it retrieves the Einstein’s gravity in the IR limit. Recently its black hole solution and thermodynamics have been intensively investigated [28, 33–36, 38–47]. By introducing a dynamical parameter  $\Lambda$  in asymptotically  $\text{AdS}_4$  space-time, a spherically symmetric black hole solution was first given by Lu et al. [28]. In this paper, motivated by all the above mentioned features, we extract the whole thermodynamic quantities of this black hole, called Lu Mei Pope (LMP) black hole in arbitrary space time. Also we study the phase structure of this solution using geometric methods as well as using the Ehrenfest scheme.

The paper is organized as follows. In section 2, we discuss the thermodynamics of LMP black hole in HL gravity and we calculate the equation of state of the black hole system. Using the idea of geometrothermodynamics, the thermodynamics and the peculiar behaviors of thermodynamic variables are studied in section 3. In section 4, using the Ehrenfest scheme, the nature of phase transition is discussed. Finally the conclusions are quoted in section 5.

## 2 Review of thermodynamics of LMP black hole

Hořava used the ADM formalism, where the four-dimensional metric of general relativity is parameterised as [29],

$$ds_4^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \quad (2.1)$$

where  $N$ ,  $N^i$  and  $g_{ij}$  are the lapse, shift and 3-metric respectively. The ADM decomposition (2.1) of the Einstein-Hilbert action is given by,

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N (K_{ij} K^{ij} - K^2 + R - 2\Lambda), \quad (2.2)$$

where  $G$  is Newton's constant,  $R$  is the curvature scalar and  $K_{ij}$  is defined by,

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (2.3)$$

The action of the theory proposed by Hořava [30] can be written as,

$$S_{\text{HL}} = \int dt d^3x \sqrt{g} N \left[ \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)} \right\} \right. \\ \left. + \left\{ \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 - \frac{\kappa^2}{2w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right) \right\} \right], \quad (2.4)$$

where  $\lambda, \kappa, \mu, w$  and  $\Lambda_W$  are constant parameters, and  $C_{ij}$  is the Cotton tensor, defined by,

$$C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R^j_{\ell} - \frac{1}{4} R \delta^j_{\ell} \right). \quad (2.5)$$

Comparing the first term in (2.4) with that of general relativity in the ADM formalism, we can write the speed of light, Newton's constant and the cosmological constant respectively as,

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{1-3\lambda}}, \quad G = \frac{\kappa^2}{32\pi c} \quad \text{and} \quad \Lambda = \frac{3}{2} \Lambda_W. \quad (2.6)$$

Now we will look for a static and spherically symmetric solution with the metric,

$$ds^2 = f(r) dt^2 - \frac{dr^2}{f(r)} + r^2 d\Omega^2. \quad (2.7)$$

In the present study we are interested in the solution with the choice  $\lambda = 1$ . This will lead to the LMP black hole solution [28], given by,

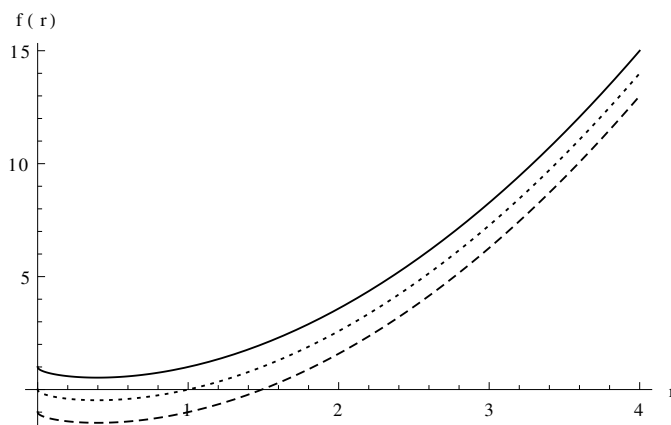
$$f(r) = k - \Lambda_W r^2 - \mathcal{A} \sqrt{\frac{r}{-\Lambda_W}}, \quad (2.8)$$

where  $\mathcal{A}$  is an integration constant and is related to the black hole mass as  $\mathcal{A} = aM$ . It is interesting to note that this solution (2.8) is asymptotically  $\text{AdS}_4$ . From the first law of black hole mechanics [31], when a black hole undergoes a change from a stationary state to another, then the change in mass of the black hole is given by,

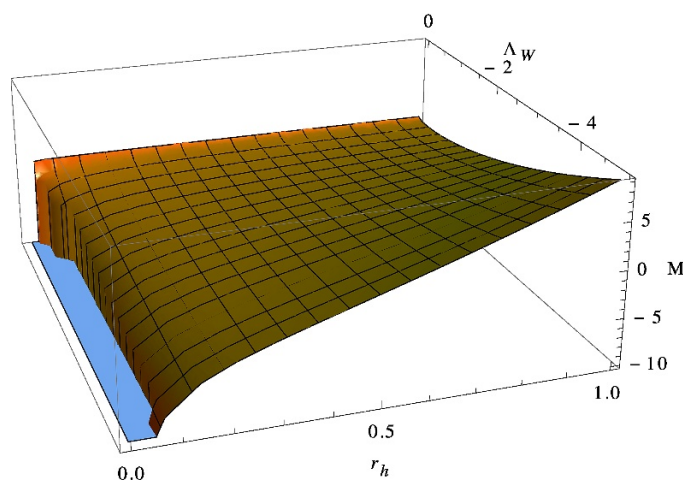
$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ. \quad (2.9)$$

Comparing this with the first law of thermodynamics,

$$dM = T dS + \Omega dJ + \Phi dQ, \quad (2.10)$$



**Figure 1.** Plots of  $f(r)$  vs  $r$  for  $k = 0$  (solid line),  $k = 1$  (dotted line) and  $k = -1$  (dashed line) with  $\Lambda_W = -1$ ,  $a = 1$ ,  $M = 1$ .



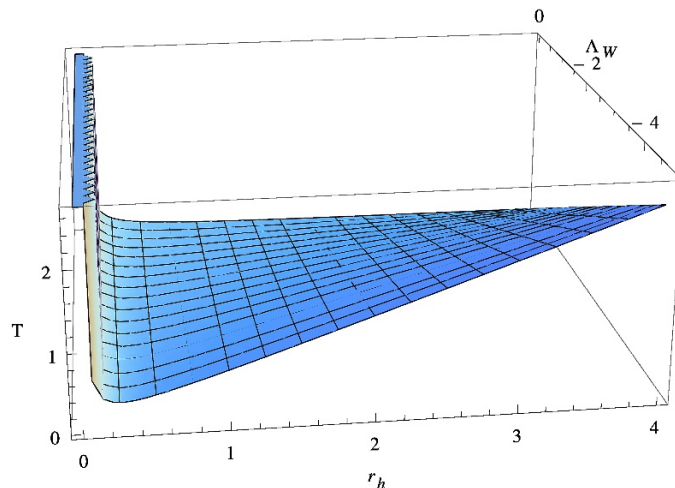
**Figure 2.** 3D Plots of Mass vs  $r_h$ .

one can easily establish the analogy between black hole mechanics and the first law of thermodynamics. We know that Hořava-Lifshitz theory does not possess the full diffeomorphism invariance of general relativity but only a subset in the form of local Galilean invariance. This subset is manifest in the Arnowitt, Deser and Misner (ADM) slicing. Here we have considered the ADM decomposition of the four dimensional metric. Then for a non-rotating uncharged black hole, the entropy can be written as [36, 37],

$$S = \int \frac{dM}{T} = \int \frac{1}{T_H} \frac{\partial H}{\partial r_h} dr_h, \quad (2.11)$$

where  $H$  denotes the enthalpy and  $r_h$  denotes the horizon radius. And the Hawking temperature can be determined from,

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} f'(r) \Big|_{r=r_h}, \quad (2.12)$$



**Figure 3.** 3D Plots of temperature.

and,

$$T_H = \left( \frac{\partial H}{\partial S} \right)_P, \quad (2.13)$$

where  $P$  is the pressure and it is related to the Hawking temperature and entropy as,

$$P = \frac{1}{2} T_H S. \quad (2.14)$$

Hence the volume of the black hole is given by,

$$V = \left( \frac{\partial H}{\partial P} \right)_S. \quad (2.15)$$

The heat capacity at constant pressure and at constant volume can be obtained respectively as,

$$C_P = T \left( \frac{\partial S}{\partial T} \right)_P, \quad (2.16)$$

and,

$$C_V = C_P + V \frac{\partial P}{\partial T}. \quad (2.17)$$

Using these relations we can calculate the thermodynamic quantities of the LMP black holes in arbitrary space curvature. In the cases of spherical ( $k = 1$ ) and flat spaces ( $k = 0$ ), detailed studies are done in [32]. And it is noted that, in both cases the black hole doesn't show any kind of phase transition behaviors. In this paper we are interested in the LMP black hole solution in Hyperbolic space ( $k = -1$ ). In this case, (2.8) can be reduced to,

$$f(r) = -1 - \Lambda_W r^2 - \mathcal{A} \sqrt{\frac{r}{-\Lambda_W}}. \quad (2.18)$$

The event horizon can be obtained from  $f(r_h) = 0$ , and from that one can easily arrive at the black hole mass-event horizon radius relation as,

$$M = \frac{1}{a} \sqrt{\frac{-\Lambda_W}{r_h}} (-1 - \Lambda_W r_h^2). \quad (2.19)$$

In figure 2 we draw the 3D plot of variation of black hole mass with respect to black hole horizon radius for varying cosmological constant term( $\Lambda_W$ ). From this figure it can be easily seen that the black hole mass increases with increase in the magnitude of the cosmological constant. It is interesting to note from this figure that the black hole vanishes for lower values of  $S$ . Considering the numerical values, say  $a = 1$  and  $\Lambda_W = -1$  the black hole vanishes at  $r_h = 1$ , but when  $\Lambda_W = -2$  the black hole vanishes at another value of horizon radius,  $r_h = 0.5$ , and so on.

Black hole entropy can be obtained from (2.11) as,

$$S = \frac{8\pi\sqrt{-\Lambda_W}}{a}. \quad (2.20)$$

From (2.12) we can derive the Hawking temperature as,

$$T_H = \frac{(1 - 3\Lambda_W r_h^2)}{8\pi r_h}, \quad (2.21)$$

3D plot of Hawking temperature with respect to black hole horizon radius for varying cosmological constant term is depicted in figure 3. Here also the temperature increases with the magnitude of the cosmological constant. From (2.14), black hole pressure can be found as

$$P = \frac{21r_h(1 - 3\Lambda_W r_h^2)}{64}. \quad (2.22)$$

From the above expression, it is obvious that for any negative value of the cosmological constant term  $\Lambda_W$ , the pressure is found to be positive. Using (2.15), black hole volume is given by,

$$V = \frac{32\sqrt{-\Lambda_W}}{21ar_h^{\frac{3}{2}}} \frac{(1 - 3\Lambda_W r_h^2)}{(1 - 9\Lambda_W r_h^2)}. \quad (2.23)$$

Using (2.16) and (2.17), the heat capacity at constant pressure and at constant volume are respectively determined as,

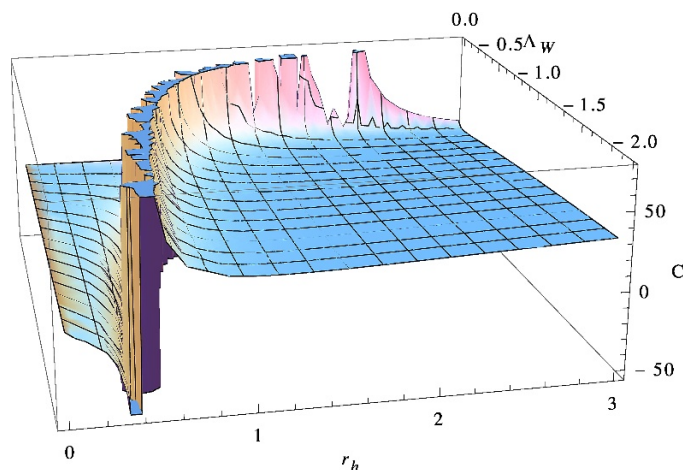
$$C_P = \frac{4\pi\sqrt{-\Lambda_W r_h}}{a} \frac{(3\Lambda_W r_h^2 - 1)}{(3\Lambda_W r_h^2 + 1)}, \quad (2.24)$$

and,

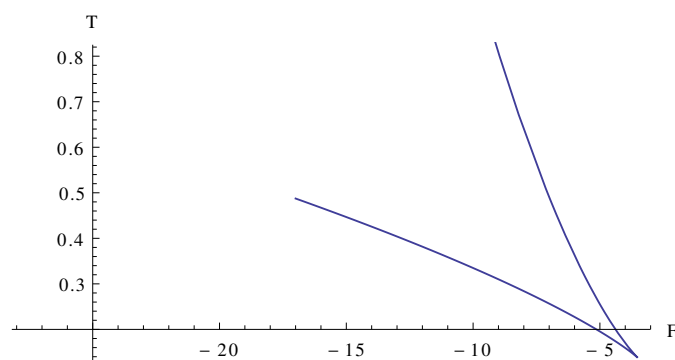
$$C_V = \frac{8\pi\sqrt{-\Lambda_W r_h}}{a} \frac{(3\Lambda_W r_h^2 - 1)}{(3\Lambda_W r_h^2 + 1)}. \quad (2.25)$$

In figure 4 we draw the 3D variations of heat capacity with respect to horizon radius and for varying cosmological constant term. From this figure, it is evident that the black hole has both positive and negative values in certain parametric regions. It is also clear from the figure that, heat capacity has a divergent point. According to Davies [88], second order phase transitions takes place at those points where the heat capacity diverges. So LMP black hole undergoes a phase transition in this case. By investigating the free energy of the black hole we can get a clear picture of the phase transition. Free energy of the black hole is given by,

$$F = M - TS. \quad (2.26)$$



**Figure 4.** 3D Plots of heat capacity.



**Figure 5.** Parametric plot of free energy and temperature for  $\Lambda_W = -1$ ,  $a = 1$ .

Using (2.19), (2.20) and (2.21) free energy of LMP black hole is obtained as,

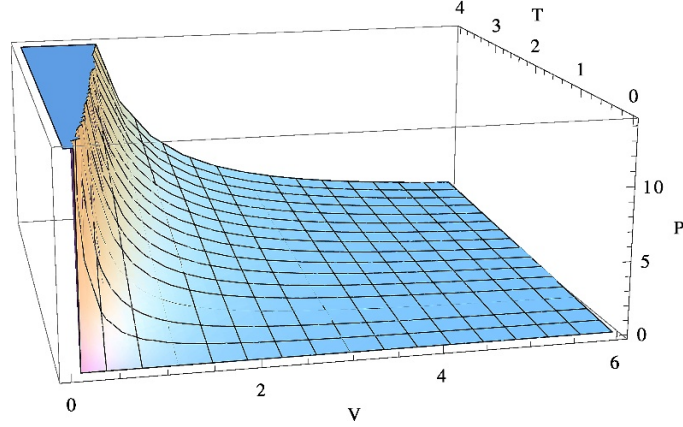
$$F = \frac{1}{a} \sqrt{\frac{-\Lambda_W}{r_h}} \left( \frac{2(1 + \Lambda_W r_h^2)(1 - 9\Lambda_W r_h^2) + (3\Lambda_W r_h^2 - 1)^2}{2(9\Lambda_W r_h^2 - 1)} \right). \quad (2.27)$$

Now one can plot the parametric variation of free energy and temperature using (2.27) and (2.21). From figure 5, it is evident that there is a cusp like double point. This indicates a second order phase transition. For one of the branches the free energy decreases and reaches the temperature which corresponds to the minimum free energy. There after, free energy increases with a different slope.

An equation of state in general, is a thermodynamic equation describing the state of matter under a given set of physical conditions. It is a constitutive equation which provides a mathematical relationship between two or more state functions associated with the matter, such as its temperature, pressure, volume, or internal energy and black hole equation of state can be written from (2.22), (2.23) and (2.21) as,

$$PV^{\frac{4}{3}} = \frac{4\pi T}{3} \sqrt{\frac{32 - \Lambda_W}{63}} \frac{1}{a^2}. \quad (2.28)$$





**Figure 6.** Isotherm  $P - V$  diagram.

Here,  $P$  is the the pressure,  $V$  is the thermodynamic volume and  $T$  is the black hole temperature. Now, we plot the isotherm  $P - V$  diagram in figure 6. From (2.28) and the corresponding figure (figure 6) we can conclude that the behaviour is “ideal gas like”. Hence no critical point can be found and there would be no  $P - V$  criticality.

### 3 Geometrothermodynamics

In order to introduce the idea of differential geometry to thermodynamics, according to [48–50], we have to define a  $(2n + 1)$  dimensional thermodynamic phase space  $\mathcal{T}$ . It can be coordinated by the set  $Z^A = \{\Phi, E^a, I^a\}$ , where  $\Phi$  represents the thermodynamic potential and  $E^a$  and  $I^a$  represent extensive and intensive thermodynamic variables respectively. The phase space is provided with the Gibbs 1-form  $\Theta = d\Phi - \delta_{ab} I^a dE^b$ , satisfying the condition,  $\Theta \wedge (d\Theta)^n \neq 0$ . Consider a Riemannian metric  $G$  on  $\mathcal{T}$ , which must be invariant with respect to Legendre transformations. Then the Riemannian contact manifold can be defined as the set  $(\mathcal{T}, \Theta, G)$ , and the equilibrium manifold can be written as a sub manifold of  $\mathcal{T}$ , i.e.,  $\mathcal{E} \subset \mathcal{T}$ . This sub manifold satisfies the condition  $\varphi^*(\Theta) = 0$  (where  $\varphi : \mathcal{E} \rightarrow \mathcal{T}$ ), known as the pull back condition [61]. Then the non-degenerate metric  $G$  and the thermodynamic metric  $g$  can be written as,

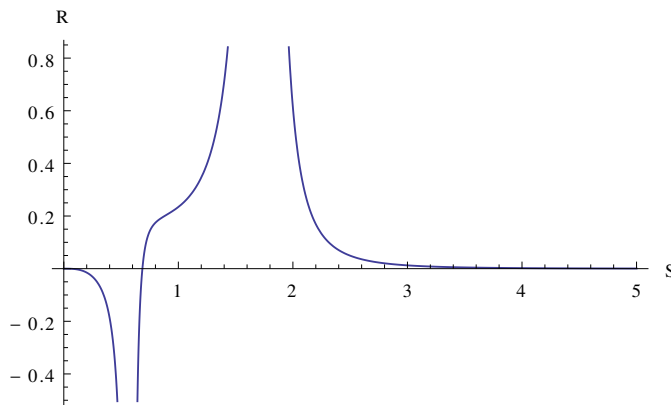
$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\eta_{cd} dE^c dI^d), \quad (3.1)$$

and,

$$g^Q = \varphi^*(G) = \left( E^c \frac{\partial \Phi}{\partial E^c} \right) \left( \eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d \right), \quad (3.2)$$

with  $\eta_{ab} = \text{diag}(-1, 1, 1, \dots, 1)$  and this metric is Legendre invariant because of the invariance of the Gibbs 1-form.

Now we will introduce this idea of Geometrothermodynamics in to the LMP black hole system to study whether the black hole exhibits a phase transition or not. For this, we will consider a 5-dimensional thermodynamic phase space  $\mathcal{T}$  constituted by extensive variables



**Figure 7.** Variation of scalar curvature with horizon radius for  $\Lambda_W = -1$ ,  $a = 1$ .

$S$  and  $a$  and the corresponding intensive variables  $T$  and  $A$ . Thus the fundamental 1-form defined on  $\mathcal{T}$  can be written as,

$$d\Theta = dM - T dS - a dA. \quad (3.3)$$

Now the Quevedo metric [48] is given by,

$$g = (SM_S + aM_a) \begin{bmatrix} -M_{SS} & 0 \\ 0 & M_{aa} \end{bmatrix}.$$

Then the Legendre invariant scalar curvature corresponding to the above metric is given by,

$$R = \frac{48a^2r^3(35r^6\Lambda_W^3 - 5r^4\Lambda_W^2 + 9r^2\Lambda_W + 9)}{(r^2\Lambda_W + 3)^2(3r^2\Lambda_W + 1)^2(5r^2\Lambda_W - 3)^3}. \quad (3.4)$$

We have plotted the variation of scalar curvature with horizon radius in figure 7. From this figure as well as from the above equation, it can be confirmed that this scalar curvature diverges at the same point where the heat capacity diverges. Hence the Geometrothermodynamics exactly reproduces the phase transition structure of the LMP black hole.

## 4 Analytical check of classical Ehrenfest equations

Infinite discontinuity in the heat capacity of the black hole does not always indicate a second order phase transition, but it suggests the possibility of a higher order phase transition. In classical thermodynamics, one can confirm the first order phase transition by utilizing Clausius-Clapeyron equations. Similarly a second order transition can be confirmed by checking whether it satisfies Ehrenfest equations or not. The original expressions of Ehrenfest equations in classical thermodynamics are given by,

$$\left(\frac{\partial P}{\partial T}\right)_S = \frac{C_{P_2} - C_{P_1}}{VT(\alpha_2 - \alpha_1)} = \frac{\Delta C_P}{VT\Delta\alpha}, \quad (4.1)$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\alpha_2 - \alpha_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta\alpha}{\Delta\kappa}, \quad (4.2)$$

where  $\alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P$  is the volume expansion coefficient and  $\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T$  is the isothermal compressibility coefficient. Considering the analogy between the thermodynamic variables and black hole parameters, where pressure ( $P$ ) is replaced by the negative of the electrostatic potential difference ( $-\Phi$ ), and volume ( $V$ ) is replaced by charge of the black hole ( $Q$ ). Thus for black hole thermodynamics, the two Ehrenfest equations (4.1) and (4.2) become,

$$-\left( \frac{\partial \Phi}{\partial T} \right)_S = \frac{1}{QT} \frac{C_{\Phi_2} - C_{\Phi_1}}{(\alpha_2 - \alpha_1)} = \frac{\Delta C_{\Phi}}{QT \Delta \alpha}, \quad (4.3)$$

$$-\left( \frac{\partial \Phi}{\partial T} \right)_Q = \frac{\alpha_2 - \alpha_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta \alpha}{\Delta \kappa}, \quad (4.4)$$

where  $\alpha = \frac{1}{Q} \left( \frac{\partial Q}{\partial T} \right)_{\Phi}$  is the volume expansion coefficient and  $\kappa_T = -\frac{1}{Q} \left( \frac{\partial Q}{\partial \Phi} \right)_T$  is the isothermal compressibility coefficient of the black hole system. Here, in the above sets of equations, the subscripts 1 and 2 denote two distinct phases of the system.

In this paper, rather than considering the black hole analogy of Ehrenfest equation, we will introduce the classical Ehrenfest equation directly in to the black hole system under consideration. Using (2.20), (2.21), (2.22) and (2.23), we can arrive at the expressions of specific heat at constant pressure, volume expansion coefficient and isothermal compressibility coefficient respectively as,

$$C_P = \frac{4\pi\sqrt{-\Lambda_W r_h}}{a} \frac{(3\Lambda_W r_h^2 - 1)}{(3\Lambda_W r_h^2 + 1)}, \quad (4.5)$$

$$\alpha = \frac{6\pi r_h}{1 - 3r_h^2 \Lambda_W}, \quad (4.6)$$

and,

$$\kappa = \frac{16}{7r_h} \frac{1}{1 - 3r_h^2 \Lambda_W}. \quad (4.7)$$

From these relations, it is interesting to note that both volume expansion coefficient and isothermal compressibility coefficient have same factor in the denominator, which implies that both these parameters diverge at the same point. We have plotted the variation of these coefficients with respect to the horizon radius ( $r_h$ ) in figures 8 and 9 respectively.

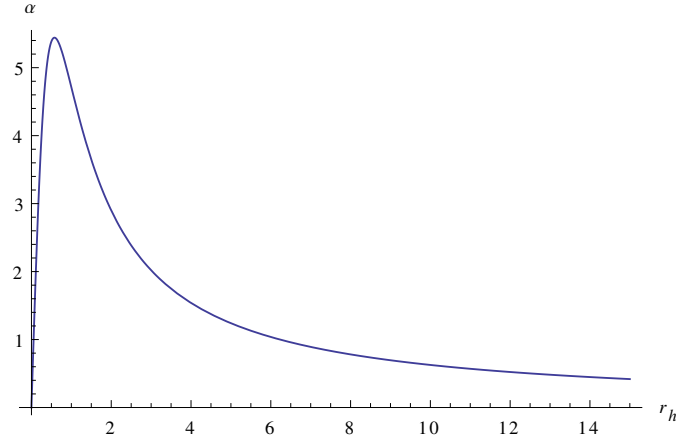
Now we will investigate the nature of phase transition at the critical point of LMP black hole by doing the analytic check of classical Ehrenfest equations (4.1) and (4.2). The values of temperature, pressure and volume at the critical point are respectively given by,

$$T_c = \frac{\sqrt{3}\sqrt{\Lambda_W}}{4\pi}, \quad (4.8)$$

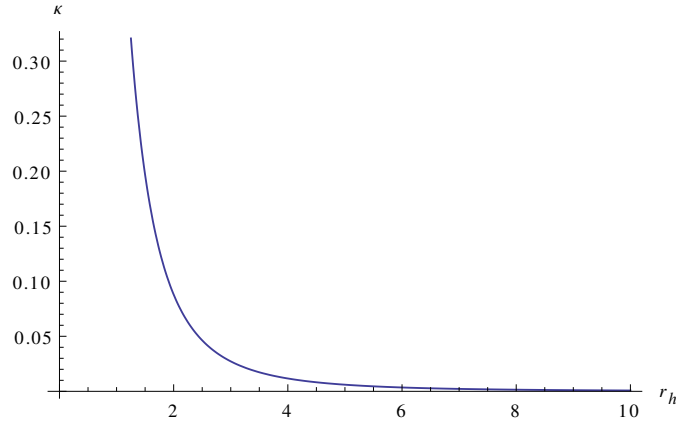
$$P_c = \frac{7\sqrt{3}}{32\sqrt{\Lambda_W}}, \quad (4.9)$$

and,

$$V_c = \frac{16(-\Lambda_W)^{5/4}}{7\sqrt[4]{3}a}. \quad (4.10)$$



**Figure 8.** Variation of volume expansion coefficient with horizon radius for  $\Lambda_W = -1$ ,  $a = 1$ .



**Figure 9.** Variation of isothermal compressibility with horizon radius for  $\Lambda_W = -1$ ,  $a = 1$ .

Now let's check the validity of Ehrenfest equations at the critical point. From the definition of volume expansion coefficient  $\alpha$ , given by (4.6), we obtain,

$$V\alpha = \left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial V}{\partial S}\right)_P \left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial V}{\partial S}\right)_P \left(\frac{C_P}{T}\right), \quad (4.11)$$

then, the r.h.s. of first classical Ehrenfest equation (4.1) becomes,

$$\frac{\Delta C_P}{TV\Delta\alpha} = \left[ \left(\frac{\partial S}{\partial V}\right)_P \right]_{r_{\text{cri}}}, \quad (4.12)$$

where  $r_{\text{cri}}$  denotes the critical point. Applying the above equation to LMP black hole system, we obtain,

$$\frac{\Delta C_P}{TV\Delta\alpha} = \frac{21\pi}{24} \frac{1}{-\Lambda_W}. \quad (4.13)$$

Now the l.h.s. of first classical Ehrenfest equation (4.1) becomes,

$$\left[ -\left(\frac{\partial P}{\partial T}\right)_S \right]_{r_{\text{cri}}} = \frac{21\pi}{24} \frac{1}{-\Lambda_W}. \quad (4.14)$$

From (4.14) and (4.13), we can arrive at the conclusion that both l.h.s. and r.h.s. of first Ehrenfest equation are in good agreement at the critical point  $r_{\text{cri}}$ . From (4.6) and (4.7), using the thermodynamic identity,

$$\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P = -1, \quad (4.15)$$

we can obtain,

$$V\kappa_T = -\left(\frac{\partial V}{\partial P}\right)_T = \left(\frac{\partial T}{\partial P}\right)_V \left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial T}{\partial P}\right)_V V\alpha. \quad (4.16)$$

Now the r.h.s. of (4.2) can be obtained as,

$$\frac{\Delta\alpha}{\Delta\kappa_T} = \left[\left(\frac{\partial P}{\partial T}\right)_V\right]_{r_{\text{cri}}} = \frac{21\pi}{24} \frac{1}{(-\Lambda_W)^{\frac{4}{3}}}. \quad (4.17)$$

Also the l.h.s. of (4.2) can be obtained as,

$$\left[\left(\frac{\partial P}{\partial T}\right)_V\right]_{r_{\text{cri}}} = \frac{21\pi}{24} \frac{1}{(-\Lambda_W)^{\frac{4}{3}}}. \quad (4.18)$$

From (4.18) and (4.17), we can conclude that second Ehrenfest equation is satisfied at the critical points. Hence both the Ehrenfest equations are in good agreement at the critical point. Using (4.13) and (4.17), the Prigogine-Defay (PD) can be obtained as

$$\Pi = \frac{\Delta C_P \Delta\kappa_T}{TV(\Delta\alpha)^2} = 1. \quad (4.19)$$

This confirms that the phase transition of LMP black hole in Hořava-Lifshitz gravity is second order in nature.

## 5 Conclusion

The complete thermodynamics and phase transition picture of LMP black holes in Hořava-Lifshitz gravity have been investigated using both thermodynamic geometry and Ehrenfest's scheme. We have systematically analyzed the thermodynamics and phase transition. From this thermodynamic study, absence of any discontinuity in entropy-temperature relationship eliminates the presence of any first order transition. Then the heat capacity is found to be diverging, thereby indicating the presence of a phase transition. But the order of phase transition was not revealed. To further clarify the existence of phase transition, geometrothermodynamics is applied, in which the critical point where the heat capacity diverges coincides with the diverging point of Legendre invariant geometrothermodynamic scalar curvature. Hence GTD metric exactly reproduces the phase transition structure of LMP black hole and their corresponding thermodynamic interactions. Here we can conclude that the curvature scalar behaves in a similar way as that of the black hole system. Then we have conducted a detailed analytic check of classical Ehrenfest equations on LMP black hole system. From this we have found that the LMP black hole satisfies the Ehrenfest

equations, and hence the phase transition is second order in nature. The PD ratio found in these calculations further witnesses the second order phase transition, and confirms that there is no deviation from this order. Hence it will be possible to answer whether one can have a quantum field theory at a finite temperature by studying the thermodynamic stability of the black hole, as evident from the specific heat. However, the black hole configuration must be favourable over pure thermal radiation in anti-de Sitter space, that is, have dominant negative free energy. The present black hole solution satisfies these conditions. Hence from this study one can look forward for the implication on the dual field theory which exists on the boundary of the anti-de Sitter space.

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